Numerical Optimal Control for Pattern Generation

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Part I

Optimal control

Problem definition

$$\min_{X,U} l_T\big(x(T)\big) + \int_0^T l\big(x(t), u(t)\big)dt$$

$$\text{s.t.} \quad \dot{x}(t) = f(x(t),u(t))$$

- X and U are functions of t:

  $X: t \in \Re \rightarrow x(t) \in \Re^{\mathrm{n}}$

- The terminal T is fixed

Problem definition

$$\min_{X,U} \; l_T(x_T) + \sum_{t=0}^{T-1} l(x_t, u_t)$$

$$\text{s.t.} \quad x(t+1) = f(x(t),u(t))$$

- X and U are vectors of dimension $T*nx$ and $T*nu$ resp.

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad U = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

- The information in X and U is somehow redundant

Problem definition

$$\min_{\cancel{X,U}} l_T(x_T) + \sum_{t=0}^{T-1} l(x_t, u_t)$$

$$\cancel{\text{s.t.} \quad x(t+1) = f(x(t),u(t))}$$

$$u(t) := f^{-1}(x(t), x(t+1))$$

Explicit formulation

Problem definition

$$\min_{\cancel{X},U} \; l_T(x_T) + \sum_{t=0}^{T-1} l(x_t, u_t)$$

$$\cancel{\text{s.t.} \quad x(t+1) = f(x(t),u(t))}$$

$$x(t) := f_t\big(x(0),\, u(0), \ldots, u(t)\big)$$

Implicit formulation

Problem definition

$$\min_{X,U} x_T{}^T L_T x_T + \sum_{t=0}^{T-1} x_t{}^T L_X x_t + u_t{}^T L_U u_t$$

$$\text{s.t.} \quad x(t+1) = F_X x(t) + F_u u(t)$$

Linear quadratic regulator

Quadratic programing

- Formulation

$$\min_x \frac{1}{2} x^T H\, x \; - g^T x$$

$$\text{s.t.} \quad A\, x = b$$

- Resolution
  - Constraint level

$$x = A^+ \, b + Z\, z \;\; , \quad Z = ker(A)$$

  - Constraint free reduction

$$\min_z \frac{1}{2} \, z^T \widetilde{H} \, z \; - \tilde{g}^T z$$

  - Minimum obtained when the derivative vanishes

$$z^* = \widetilde{H}^{-1} \, \tilde{g}$$

Part II

Pattern generator

Table-cart model

- Notation

$c=(c^x,c^y,c^z)$	Center of Mass
m	Total mass of the system
g	Gravity
f_i , p_i	Force fi applied at contact point pi
L	Angular momentum ("*moment cinétique*" in French) expressed at the center of mass
z	Center of pressure

Table-cart model

- Newton

$$m(\ddot{c} + g) = \sum f_i \qquad (1)$$

- Euler

$$\dot{L} = \sum (p_i - c) \times f_i \qquad (2)$$

- Summing c × (1) with (2)

$$m\, c \times (\ddot{c} + g) + \dot{L} = \sum p_i \times f_i \qquad (3)$$

- Then for $p_i^z = 0,\ g_i^x = 0,\ g_i^y = 0$:

$$mc^y(\ddot{c}^z + g) - mc^z\ddot{c}^y + \dot{L}^x = \sum p_i^x f_i^z$$

$$mc^x(\ddot{c}^z + g) - mcz\ddot{c}^x - \dot{L}^y = \sum p_i^y f_i^z$$

Table-cart model

$$mc^y(\ddot{c}^z + g) - mc^z\ddot{c}^y + \dot{L}^x = \sum p_i^x f_i^z$$

$$mc^x(\ddot{c}^z + g) - mc^z\ddot{c}^x - \dot{L}^y = \sum p_i^y f_i^z$$

- And finally, dividing by $m(\ddot{c}^z + g) = \sum f_i^z$

$$c^x - \frac{mc^z\ddot{c}^x - \dot{L}^y}{m(\ddot{c}^z + g)} = \frac{\sum p_i^x f_i^z}{\sum f_i^z}$$

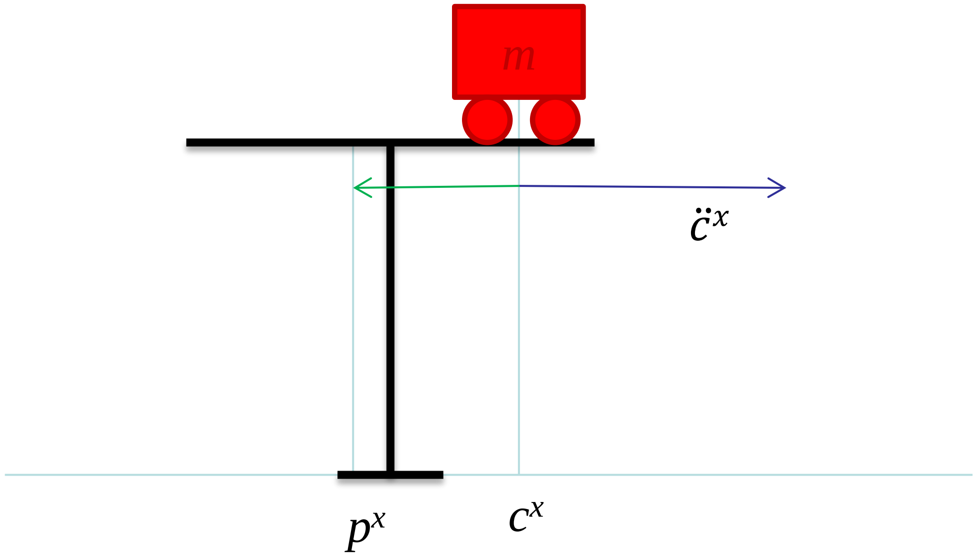
$$c^y - \frac{mc^z\ddot{c}^y + \dot{L}^x}{m(\ddot{c}^z + g)} = \frac{\sum p_i^x f_i^z}{\sum f_i^z}$$

- For negligible $\ddot{c}^z$ and $\dot{L}$:

$$\boxed{c^{xy} - \frac{1}{\omega^2}\ddot{c}^{xy} = z^{xy}} \quad , \qquad \text{with } \omega = \sqrt{\frac{g}{c^z}}$$

Table-cart model

$$\boxed{c^{xy} - \frac{1}{\omega^2}\ddot{c}^{xy} = z^{xy}} \quad , \qquad \text{with } \omega = \sqrt{\frac{g}{c^z}}$$



Pattern generator problem

- Variables:
  - State : $x_t = (c^x, c^y, \dot{c}^x, \dot{c}^y, \ddot{c}^x, \ddot{c}^y)$
  - Control : $u_t = (\dddot{c}^x, \dddot{c}^y)$
- Cost

$$\sum_{t=0}^{T-1} ||u(t)||^2 + ||c^{xy}(t) - \frac{1}{\omega^2}\,\ddot{c}^{xy}(t) - z^{xy\,*}\,||^2$$

- Constraint:

$$c^{xy}(t+1) = c^{xy}(t) + \Delta t\,\dot{c}^{xy}(t)$$
$$\dot{c}^{xy}(t+1) = \dot{c}^{xy}(t) + \Delta t\,\ddot{c}^{xy}(t)$$
$$\ddot{c}^{xy}(t+1) = \ddot{c}^{xy}(t) + \Delta t\,u^{xy}(t)$$

And now ?

- From z^{xy*}, compute an optimum c^{xy} trajectory
  - Build z^{xy*} from the footprints
  - Implement the QP problem
  - Solve it using the pseudo inverse

- Compute some foot trajectories
  - From the footprints
  - First approximation: sliding the feet on the ground

- Compute the whole-body movement
  - Using inverse kinematics
  - Main tasks: COM, right and left feet
  - Some other tasks to constraint the overall posture